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A Relationship Between the Dirac and Maxwell Equations

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Abstract

It is shown that an analogue of Dirac's equation may be constructed from Milner's extension of Maxwell's equations and a comparison of the algebraic and physical features of the equations made.

In a recent paper, Muraskin (1970) has examined the relationships between Dirac's and Maxwell's equations in a tensorial representation. Their relationship was studied extensively by the late Professor Milner (1936, 1961, 1963), who was especially concerned to adopt a physicist's rather than a mathematician's approach. For this reason he worked with column-matrices in flat Euclidean space, and it may be of interest to record that recent developments of his theories have led to a particularly simple analogue of Dirac's equation in terms of classical electromagnetic theory.

Dirac's equation is, of course, a factorisation of the Klein–Gordon wave equation, which may be written

$$\Box^2 \psi = k^2 \psi \qquad (1)$$

where $\Box^2$ is the D'Alembertian and k^2 a constant, and Dirac's first-order equation may be written (Richards, 1959).

$$i\frac{\partial\psi}{\partial t} + i\boldsymbol{\alpha}\,.\,c\,\mathrm{div}\boldsymbol{\psi} - \frac{\beta}{\hbar} m_0 c^2 = 0 \qquad (2)$$

where

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 2\delta_{ij} \qquad (\alpha_0 = \beta; i,j = 0,1,2,3)$$

Starting from an energy-stress tensor composed of the elements of the column-matrices e and h, where $e = (e_t, e_x, e_x, e_z)$ and $h = (h_t, h_x, h_y, h_z)$, and using the equation

$$\partial_i Z_{ik} = 0 \qquad (3)$$

where Z is the energy-stress tensor, Milner (1961, 1963) obtains the equation

$$\bar{\partial} R^{-1}(e + ih) = r + is \qquad (4)$$

This has been written in Milner's notation for ease of reference, but for our present purposes it is only necessary to observe that the operator $\bar{\partial}R^{-1}$ represents quaternionic differentiation.

Equation (4) when written out in full becomes

$$\left(\operatorname{div}\mathbf{e} - r_t + \frac{1}{c}\frac{\partial e_t}{\partial t}\right) + i\left(\operatorname{div}\mathbf{h} - s_t + \frac{1}{c}\frac{\partial h_t}{\partial t}\right) = 0$$

$$i\left(\frac{1}{c}\frac{\partial \mathbf{e}}{\partial t} - \operatorname{curl}\mathbf{h} + \mathbf{r} + \operatorname{grad} e_t\right) - \left(\frac{1}{c}\frac{\partial h}{\partial t} + \operatorname{curl}\mathbf{e} + \mathbf{s} + \operatorname{grad} h\right) = 0 \qquad (5)$$

Rearranging, introducing the electric and magnetic charge and current densities and taking resolutes, leads to

$$\operatorname{div}\mathbf{e} = j_t = r_t - \frac{1}{c}\frac{\partial e_t}{\partial t} \qquad (6a)$$

$$\frac{1}{c}\frac{\partial \mathbf{e}}{\partial t} - \operatorname{curl}\mathbf{h} = -\mathbf{j} = \mathbf{r} - \operatorname{grad} e_t \qquad (6b)$$

$$\operatorname{div}\mathbf{h} = k_t = s_t - \frac{1}{c}\frac{\partial h_t}{\partial t} \qquad (6c)$$

$$\frac{1}{c}\frac{\partial h}{\partial t} + \operatorname{curl}\mathbf{e} = -\mathbf{k} = s - \operatorname{grad} h_t \qquad (6d)$$

Taking the first equality in each equation gives the set of Maxwell's equations. The second equalities show that in Milner's theory the charge and current densities are composed of two quantities, one of which only is the source term. It is this fact that leads to the analogue with Dirac's equation.

Milner (1961, 1963) argued for the additional relations

$$r = \kappa h \qquad \text{and} \qquad s = \kappa e \qquad (7)$$

where κ is a constant.

However, as Kilmister (1963) pointed out, κ is really a four-vector, and the writer has substituted the relation

$$r = \eta\kappa h \qquad \text{and} \qquad s = \eta\kappa e \qquad (8)$$

where

$$\eta = \begin{matrix} 1 & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & -1 \end{matrix}$$

which effectively substitutes κh_t for r_t, $-\kappa\mathbf{h}$ for $\mathbf{r}$, κe_t for s_t and $-\kappa\mathbf{e}$ for $\mathbf{s}$ in equations (5) and (6).

It is then easily demonstrated that all components of (5) and (6) obey a wave equation of the form

$$\Box^2\chi = -\kappa^2\chi \qquad (9)$$

This wave equation differs from the Klein–Gordon equation in the signature of the source term. Furthermore, this difference cannot be removed by making κ complex (Milner, 1961, 1963).

Physically, it is apparent that the two equations are essentially different, in that in the Klein–Gordon equation the wave velocity is greater, and in Milner's less, than the velocity of light c.

Returning now to equations (5) and (6) it may be noted that whereas the terms on the right-hand side and in the middle are invariant whether the motion is wave motion or not, the terms on the right-hand side interchange between (a) and (c) and (b) and (d). Thus the appropriate first-order equation in h_t that will result from substitution of a solution of the form $A\exp[i(\kappa_1 vt - \kappa_1 x)]$ will be

$$i\frac{\partial h_t}{\partial t} + ic\,\mathrm{div}\,\mathbf{h} - \kappa c h_t = 0 \tag{10}$$

The second order wave equation in h_t is

$$\nabla^2 h_t - \frac{1}{c^2}\frac{\partial^2 h_t}{\partial t^2} + \kappa^2 h_t = 0 \tag{11}$$

Substituting for h_t we have

$$-\kappa_1{}^2 + \kappa_1{}^2\frac{v^2}{c^2} + \kappa^2 = 0 \tag{12}$$

or

$$\frac{1}{\kappa_1{}^2} = \frac{1}{\kappa^2}\left(1 - \frac{v^2}{c^2}\right) \tag{13}$$

Now, Milner shows that the rest energy of a spherical charge system is proportional to κ, so (13) accords with the equation

$$m = \frac{m_0}{[1 - (v^2/c^2)]^{1/2}} \tag{14}$$

If we quantise by taking the total energy to be $\hbar\omega = \hbar\kappa_1 c = mc^2$, we have

$$\kappa = \frac{m_0 c}{\hbar} \tag{15}$$

Substituting in (10) we have

$$i\frac{\partial h_t}{\partial t} + ic\,\mathrm{div}\,\mathbf{h} - \hbar m_0 c^2 h_t = 0 \tag{16}$$

It may now be seen that (16) and (2) differ algebraically by the presence of the factor $\boldsymbol{\alpha}$ in the latter. Physically, of course, h_t is an actual charge density (in Milner's, not Maxwell's, system) whereas ψ is a probability function.

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